

Determinantal representations of the quaternion weighted Moore-Penrose inverse and corresponding Cramer's rule.

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Abstract

Weighted singular value decomposition (WSVD) and a representation of the weighted Moore-Penrose inverse of a quaternion matrix by WSVD have been derived. Using this representation, limit and determinantal representations of the weighted Moore-Penrose inverse of a quaternion matrix have been obtained within the framework of the theory of the noncommutative column-row determinants. By using the obtained analogs of the adjoint matrix, we get the Cramer rules for the weighted Moore-Penrose solutions of left and right systems of quaternion linear equations.

1 Introduction

Let \mathbb{R} and \mathbb{C} be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$, and by $\mathbb{H}_r^{m \times n}$ the set of all $m \times n$ matrices over \mathbb{H} with a rank r . Let $M(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices and \mathbf{I} be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n \times m}$, we denote by \mathbf{A}^* , rank \mathbf{A} the conjugate transpose (Hermitian adjoint) matrix and the rank of \mathbf{A} . The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices can be extended to quaternion matrices.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m \times n}$, denoted by \mathbf{A}^\dagger , is the unique matrix $\mathbf{X} \in \mathbb{H}^{n \times m}$ satisfying the following equations [1],

$$\mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{A}; \tag{1}$$

$$\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X}; \tag{2}$$

$$(\mathbf{A}\mathbf{X})^* = \mathbf{A}\mathbf{X}; \tag{3}$$

$$(\mathbf{X}\mathbf{A})^* = \mathbf{X}\mathbf{A}. \tag{4}$$

Let Hermitian positive definite matrices \mathbf{M} and \mathbf{N} of order m and n , respectively, be given. For $\mathbf{A} \in \mathbb{H}^{m \times n}$, the **weighted Moore-Penrose inverse** of \mathbf{A} is the unique solution $\mathbf{X} = \mathbf{A}_{M,N}^+$ of the matrix equations (1) and (2) and the following equations in \mathbf{X} [2]:

$$(3M) (\mathbf{M}\mathbf{A}\mathbf{X})^* = \mathbf{M}\mathbf{A}\mathbf{X}; \quad (4N) (\mathbf{N}\mathbf{X}\mathbf{A})^* = \mathbf{N}\mathbf{X}\mathbf{A}.$$

In particular, when $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{N} = \mathbf{I}_n$, the matrix \mathbf{X} satisfying the equations (1), (2), (3M), (4N) is the Moore-Penrose inverse \mathbf{A}^\dagger .

It is known various representations of the weighted Moore-Penrose. In particular, limit representations have been considered in [3, 4]. Determinantal representations of the complex (real) weighted Moore-Penrose have been derived by full-rank factorization in [5], by limit representation in [6] using the method first introduced in [7], and by minors in [8]. A basic method for finding the Moore-Penrose inverse is based on the singular value decomposition (SVD). It is available for quaternion matrices, (see, e.g. [9, 10]). In [10, 11], using SVD of quaternion matrices, the limit and determinantal representations of the Moore-Penrose inverse over the quaternion skew field have been obtained within the framework of the theory of the noncommutative column-row determinants that have been introduced in [12].

The weighted Moore-Penrose inverse $\mathbf{A}_{M,N}^\dagger \in \mathbb{C}^{m \times n}$ can be explicitly expressed by the weighted singular value decomposition (WSVD) which at first has been obtained in [13] by Cholesky factorization. In [14] WSVD of real matrices with singular weights has been derived using weighted orthogonal matrices and weighted pseudoorthogonal matrices.

Song et al. [15, 16] have studied the weighted Moore-Penrose inverse over the quaternion skew field and obtained its determinantal representation within the framework of the theory of the column-row determinants. But WSVD of quaternion matrices has not been considered and for obtaining a determinantal representation there was used auxiliary matrices which different from \mathbf{A} , and weights \mathbf{M} and \mathbf{N} .

The main goals of the paper are introducing WSVD of quaternion matrices and representation of the weighted Moore-Penrose inverse over the quaternion skew field by WSVD, and then by using this representation, obtaining its limit and determinantal representations.

In this paper we shall adopt the following notation.

Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By \mathbf{A}_β^α denote the submatrix of \mathbf{A} determined by the rows indexed by α , and the columns indexed by β . Then, \mathbf{A}_α^α denotes the principal submatrix determined by the rows and columns indexed by α . If $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}_\alpha^\alpha|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, denote by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 \leq \dots \leq \alpha_k \leq n\}$ the collection of strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let

$$I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}, \quad J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}.$$

The paper is organized as follows. We start with some basic concepts and results from the theory of the row-column determinants and of Hermitian quaternion matrices in Section 2. Weighted singular value decomposition and a representation of the weighted Moore-Penrose inverse of quaternion matrices by WSVD have been considered in Subsection 3.1, and its limit representations in

Subsection 3.2. In Section 4, we give the determinantal representations of the weighted Moore-Penrose inverse. In Subsection 4.1, if the matrices $\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\mathbf{A}$ and $\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$ are Hermitian, and if they are non-Hermitian in Subsection 4.2. In Section 5 we obtain explicit representation formulas of the weighted Moore-Penrose solutions (analogs of Cramer's rule) of the left and right systems of linear equations over the quaternion skew field. In Section 5, we give a numerical example to illustrate the main result.

2 Preliminaries

For a quadratic matrix $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ can be define n row determinants and n column determinants as follows.

Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$.

Definition 2.1 [11] *The i -th row determinant of $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is defined for all $i = \overline{1, n}$ by putting*

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}},$$

$$\sigma = (i i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$

with conditions $i_{k_2} < i_{k_3} < \dots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Definition 2.2 [11] *The j -th column determinant of $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is defined for all $j = \overline{1, n}$ by putting*

$$\text{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j k_r} a_{j k_r+l_r} \dots a_{j k_r+1 i_{k_r}} \dots a_{j j_{k_1}+l_1} \dots a_{j_{k_1}+1 j_{k_1}} a_{j_{k_1} j},$$

$$\tau = (j_{k_r}+l_r \dots j_{k_r+1} j_{k_r}) \dots (j_{k_2}+l_2 \dots j_{k_2+1} j_{k_2}) (j_{k_1}+l_1 \dots j_{k_1+1} j_{k_1} j),$$

with conditions, $j_{k_2} < j_{k_3} < \dots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Suppose \mathbf{A}^{ij} denotes the submatrix of \mathbf{A} obtained by deleting both the i th row and the j th column. Let $\mathbf{a}_{\cdot j}$ be the j -th column and $\mathbf{a}_{i \cdot}$ be the i -th row of \mathbf{A} . Suppose $\mathbf{A}_{\cdot j}(\mathbf{b})$ denotes the matrix obtained from \mathbf{A} by replacing its j -th column with the column \mathbf{b} , and $\mathbf{A}_{i \cdot}(\mathbf{b})$ denotes the matrix obtained from \mathbf{A} by replacing its i -th row with the row \mathbf{b} . We note some properties of column and row determinants of a quaternion matrix $\mathbf{A} = (a_{ij})$, where $i \in I_n$, $j \in J_n$ and $I_n = J_n = \{1, \dots, n\}$.

Proposition 2.1 [11] *If $b \in \mathbb{H}$, then $\text{rdet}_i \mathbf{A}_{i \cdot} (b \cdot \mathbf{a}_{i \cdot}) = b \cdot \text{rdet}_i \mathbf{A}$ and $\text{cdet}_i \mathbf{A}_{\cdot i} (\mathbf{a}_{\cdot i} b) = \text{cdet}_i \mathbf{A} b$ for all $i = \overline{1, n}$.*

Proposition 2.2 [11] *If for $\mathbf{A} \in M(n, \mathbb{H})$ there exists $t \in I_n$ such that $a_{tj} = b_j + c_j$ for all $j = \overline{1, n}$, then*

$$\text{rdet}_i \mathbf{A} = \text{rdet}_i \mathbf{A}_t \cdot (\mathbf{b}) + \text{rdet}_i \mathbf{A}_t \cdot (\mathbf{c}),$$

$$\text{cdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}_t \cdot (\mathbf{b}) + \text{cdet}_i \mathbf{A}_t \cdot (\mathbf{c}),$$

where $\mathbf{b} = (b_1, \dots, b_n)$, $\mathbf{c} = (c_1, \dots, c_n)$ and for all $i = \overline{1, n}$.

Proposition 2.3 [11] *If for $\mathbf{A} \in M(n, \mathbb{H})$ there exists $t \in J_n$ such that $a_{it} = b_i + c_i$ for all $i = \overline{1, n}$, then*

$$\begin{aligned} \text{rdet}_j \mathbf{A} &= \text{rdet}_j \mathbf{A}_{.t}(\mathbf{b}) + \text{rdet}_j \mathbf{A}_{.t}(\mathbf{c}), \\ \text{cdet}_j \mathbf{A} &= \text{cdet}_j \mathbf{A}_{.t}(\mathbf{b}) + \text{cdet}_j \mathbf{A}_{.t}(\mathbf{c}), \end{aligned}$$

where $\mathbf{b} = (b_1, \dots, b_n)^T$, $\mathbf{c} = (c_1, \dots, c_n)^T$ and for all $j = \overline{1, n}$.

Remark 2.1 Let $\text{rdet}_i \mathbf{A} = \sum_{j=1}^n a_{ij} \cdot R_{ij}$ and $\text{cdet}_j \mathbf{A} = \sum_{i=1}^n L_{ij} \cdot a_{ij}$ for all $i, j = \overline{1, n}$, where by R_{ij} and L_{ij} denote the right and left ij -th cofactor of $\mathbf{A} \in M(n, \mathbb{H})$, respectively. It means that $\text{rdet}_i \mathbf{A}$ can be expand by right cofactors along the i -th row and $\text{cdet}_j \mathbf{A}$ can be expand by left cofactors along the j -th column, respectively, for all $i, j = \overline{1, n}$.

The following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1 [11] *If $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $\text{rdet}_1 \mathbf{A} = \dots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \dots = \text{cdet}_n \mathbf{A} \in \mathbb{R}$.*

Since all column and row determinants of a Hermitian matrix over \mathbb{H} are equal, we can define the determinant of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$. By definition, we put

$$\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}, \quad (5)$$

for all $i = \overline{1, n}$. By using its row and column determinants the determinant of a quaternion Hermitian matrix has properties similar to a usual determinant of a real matrix. These properties are completely explored in [11] and can be summarized in the following theorems.

Theorem 2.2 *If the i -th row of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}$, where $c_l \in \mathbb{H}$ for all $l = \overline{1, k}$ and $\{i, i_l\} \subset I_n$, then*

$$\text{rdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}) = \text{cdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}) = 0.$$

Theorem 2.3 *If the j -th column of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k$, where $c_l \in \mathbb{H}$ for all $l = \overline{1, k}$ and $\{j, j_l\} \subset J_n$, then*

$$\text{cdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) = \text{rdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) = 0.$$

Theorem 2.4 [11] *If an arbitrary column of $\mathbf{A} \in \mathbf{H}^{m \times n}$ is a right linear combination of its other columns, or an arbitrary row of \mathbf{A}^* is a left linear combination of its others, then $\det \mathbf{A}^* \mathbf{A} = 0$.*

Moreover, we have the criterion of nonsingularity of a Hermitian matrix.

Theorem 2.5 [11] *The right-linearly independence of columns of $\mathbf{A} \in \mathbf{H}^{m \times n}$ or the left-linearly independence of rows of \mathbf{A}^* is the necessary and sufficient condition for $\det \mathbf{A}^* \mathbf{A} \neq 0$.*

The following theorem about determinantal representation of an inverse matrix of Hermitian follows immediately from these properties.

Theorem 2.6 [11] *If a Hermitian matrix $\mathbf{A} \in \mathbf{M}(n, \mathbb{H})$ is such that $\det \mathbf{A} \neq 0$, then there exist a unique right inverse matrix $(R\mathbf{A})^{-1}$ and a unique left inverse matrix $(L\mathbf{A})^{-1}$, and $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$, which possess the following determinantal representations:*

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{pmatrix}, \quad (6)$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{pmatrix}. \quad (7)$$

Here R_{ij} , L_{ij} are right and left ij -th cofactors of \mathbf{A} respectively for all $i, j = \overline{1, n}$.

Due to the noncommutativity of quaternions, there are two types of eigenvalues.

Definition 2.3 *Let $\mathbf{A} \in \mathbf{M}(n, \mathbb{H})$. A quaternion λ is said to be a right eigenvalue of \mathbf{A} if $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda$ for some nonzero quaternion column-vector \mathbf{x} . Similarly λ is a left eigenvalue if $\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$.*

The theory on the left eigenvalues of quaternion matrices has been investigated, in particular, in [17–19]. The theory on the right eigenvalues of quaternion matrices is more developed, for further details one may refer to [9, 20, 21]. From this theory we cite the following propositions.

Proposition 2.4 [9] *Let $\mathbf{A} \in \mathbf{M}(n, \mathbb{H})$ is Hermitian. Then \mathbf{A} has exactly n real right eigenvalues.*

Definition 2.4 *Suppose $\mathbf{U} \in \mathbf{M}(n, \mathbb{H})$ and $\mathbf{U}^* \mathbf{U} = \mathbf{U} \mathbf{U}^* = \mathbf{I}$, then the matrix \mathbf{U} is called unitary.*

Proposition 2.5 [9] *Let $\mathbf{A} \in \mathbf{M}(n, \mathbb{H})$ be given. Then, \mathbf{A} is Hermitian if and only if there are a unitary matrix $\mathbf{U} \in \mathbf{M}(n, \mathbb{H})$ and a real diagonal matrix $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{U}^*$, where $\lambda_1, \dots, \lambda_n$ are right eigenvalues of \mathbf{A} .*

Suppose $\mathbf{A} \in \mathbf{M}(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda \mathbf{I} - \mathbf{A}$ is Hermitian.

Definition 2.5 If $t \in \mathbb{R}$, then for a Hermitian matrix \mathbf{A} the polynomial $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$ is said to be the characteristic polynomial of \mathbf{A} .

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well.

Definition 2.6 Let $\mathbf{A} \in \mathbb{H}^{n \times n}$ be a Hermitian matrix and $\pi(\mathbf{A}) = \pi$ be the number of positive eigenvalues of \mathbf{A} , $\nu(\mathbf{A}) = \nu$ be the number of negative eigenvalues of \mathbf{A} , and $\delta(\mathbf{A}) = \delta$ be the number of zero eigenvalues of \mathbf{A} . Then the ordered triple $\omega = (\pi, \nu, \delta)$ will be called the inertia of \mathbf{A} . We shall write $\omega = \text{In } \mathbf{A}$.

Definition 2.7 A Hermitian matrix $\mathbf{A} \in \mathbb{H}^{n \times n}$, is called positive (semi)definite if $\mathbf{x}^* \mathbf{A} \mathbf{x} > 0 (\geq 0)$ for any nonzero vector $\mathbf{x} \in \mathbb{H}^n$.

The following properties are equivalent to \mathbf{A} being positive definite and they can be expanded obviously to quaternion matrices.

Proposition 2.6 All its eigenvalues are positive.

Proposition 2.7 Its leading principal minors are all positive.

Since all leading principal submatrices of a Hermitian matrix are Hermitian, then we can define leading principal minors as determinants of Hermitian submatrices in terms of Eq.(5).

Every positive definite matrix $\mathbf{A} \in \mathbb{H}^{n \times n}$ has a unique square root defined by $\mathbf{A}^{\frac{1}{2}}$. It means, if $\mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{U}^*$ then $\mathbf{A}^{\frac{1}{2}} = \mathbf{U} \mathbf{D}^{\frac{1}{2}} \mathbf{U}^*$.

We have [9,10] the following theorem about the singular value decomposition (SVD) of quaternion matrices.

Theorem 2.7 (SVD) Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$. There exist unitary quaternion matrices $\mathbf{V} \in \mathbb{H}^{m \times m}$ and $\mathbf{W} \in \mathbb{H}^{n \times n}$ such that $\mathbf{A} = \mathbf{V} \mathbf{\Sigma} \mathbf{W}^*$, where $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{H}_r^{m \times n}$, and $\mathbf{D}_r = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, and σ_i^2 is the nonzero eigenvalues of $\mathbf{A}^* \mathbf{A}$ for all $i = 1, \dots, r$. Then $\mathbf{A}^\dagger = \mathbf{W} \mathbf{\Sigma}^\dagger \mathbf{V}^*$, where $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

In [11], using the singular value decomposition of quaternion matrices, the limit and determinantal representations of the Moore-Penrose inverse over the quaternion skew field have been obtained as follows.

Lemma 2.1 [11] If $\mathbf{A} \in \mathbb{H}^{m \times n}$ and \mathbf{A}^\dagger is its Moore-Penrose inverse, then $\mathbf{A}^+ = \lim_{\alpha \rightarrow 0} \mathbf{A}^* (\mathbf{A} \mathbf{A}^* + \alpha \mathbf{I})^{-1} = \lim_{\alpha \rightarrow 0} (\mathbf{A}^* \mathbf{A} + \alpha \mathbf{I})^{-1} \mathbf{A}^*$, where $\alpha \in \mathbb{R}_+$.

Theorem 2.8 [11] If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $\mathbf{A}^\dagger = (a_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ possess the following determinantal representations:

$$a_{ij}^\dagger = \frac{\sum_{\beta \in J_{r, n} \setminus \{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^*))_{\beta}^{\beta}}{\sum_{\beta \in J_{r, n}} |(\mathbf{A}^* \mathbf{A})_{\beta}^{\beta}|}, \quad (8)$$

or

$$a_{ij}^\dagger = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j \cdot}(\mathbf{a}_i^*))}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*)_{\alpha}^\alpha|}. \quad (9)$$

Lemma 2.2 [11] Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ are both positive (semi)definite, and r nonzero eigenvalues of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$ coincide.

Proof. The proof of the second part immediately follows from the singular value decomposition of $\mathbf{A} \in \mathbb{H}_r^{m \times n}$. ■

Lemma 2.3 [11] If $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian, then $p_{\mathbf{A}}(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \dots + (-1)^n d_n$, where d_k is the sum of principle minors of \mathbf{A} of order k , $1 \leq k < n$, and $d_n = \det \mathbf{A}$.

Definition 2.8 A square matrix $\mathbf{Q} \in \mathbb{H}^{m \times m}$ is called **H-weighted unitary** (unitary with weight \mathbf{H}) if $\mathbf{Q}^*\mathbf{H}\mathbf{Q} = \mathbf{I}_m$, where \mathbf{I}_m is the identity matrix.

The following well-known two facts (see, e.g. [22]) on positive definite and Hermitian matrices and their product obviously can be extended to quaternion matrices.

Lemma 2.4 Let $\mathbf{A} \in \mathbb{H}^{n \times n}$ be positive definite and $\mathbf{B} \in \mathbb{H}^{n \times n}$ be Hermitian matrices, respectively. Then $\mathbf{A}\mathbf{B}$ is a diagonalizable matrix, it's all eigenvalues are real, and $\text{In}\mathbf{A}\mathbf{B} = \text{In}\mathbf{A}$.

Lemma 2.5 Let $\mathbf{A} \in \mathbb{H}^{n \times n}$ be positive definite and $\mathbf{B} \in \mathbb{H}^{n \times n}$ be Hermitian matrices, respectively. Then there exists nonsingular $\mathbf{C} \in \mathbb{H}^{n \times n}$ such that $\mathbf{C}^*\mathbf{A}\mathbf{C} = \mathbf{I}_n$, and $\mathbf{C}^*\mathbf{B}\mathbf{C} = \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is a diagonal matrix.

3 Weighted singular value decomposition and representations of the weighted Moore-Penrose inverse of quaternion matrices

3.1 Representations of the weighted Moore-Penrose inverse of quaternion matrices by WSVD

Denote $\mathbf{A}^\# = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$. Now, we prove the following theorem about the weighted singular value decomposition (WSVD) of quaternion matrices. We give the proof of the theorem that different from analogous for real matrices in [13], and this method has more similar manner to [14], where WSVD of $\mathbf{A} \in \mathbb{R}^{m \times n}$ with positive definite weights \mathbf{B} and \mathbf{C} has been described as $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{C}$.

Theorem 3.1 Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, and \mathbf{M} and \mathbf{N} be positive definite matrices of order m and n , respectively. Then there exist $\mathbf{U} \in \mathbb{H}^{m \times m}$, $\mathbf{V} \in \mathbb{H}^{n \times n}$ satisfying $\mathbf{U}^*\mathbf{M}\mathbf{U} = \mathbf{I}_m$ and $\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$ such that

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*, \quad (10)$$

where $\mathbf{D} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\boldsymbol{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and σ_i^2 is the nonzero eigenvalues of $\mathbf{A}^\sharp \mathbf{A}$ or $\mathbf{A} \mathbf{A}^\sharp$, which coincide.

Proof. First, consider $\mathbf{A}^\sharp \mathbf{A} = \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A}$. Since $\mathbf{A}^* \mathbf{M} \mathbf{A} = (\mathbf{M}^{\frac{1}{2}} \mathbf{A})^* \mathbf{A} \mathbf{M}^{\frac{1}{2}}$, then, by Lemma 2.2, $\mathbf{A}^* \mathbf{M} \mathbf{A}$ is Hermitian positive semidefinite, and by Lemma 2.4 all eigenvalues of $\mathbf{A}^\sharp \mathbf{A}$ are nonnegative. Denote them by σ_i^2 , where $\sigma_1 \geq \dots \geq \sigma_n \geq 0$.

Denote $\mathbf{L} = \mathbf{A}^\sharp \mathbf{A}$. Since $\mathbf{L} \mathbf{N}^{-1} = \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-1}$ is Hermitian and there exists a nonsingular $\mathbf{V} \in \mathbb{H}^{n \times n}$ such that $\mathbf{V}^* \mathbf{N}^{-1} \mathbf{V} = \mathbf{I}_n$, then by Lemma 2.5,

$$\mathbf{V}^* \mathbf{L} \mathbf{N}^{-1} \mathbf{V} = \boldsymbol{\Lambda}, \quad (11)$$

where \mathbf{V} is unitary with weight \mathbf{N}^{-1} , and $\boldsymbol{\Lambda}$ is a diagonal matrix.

It follows from $\mathbf{L} = \mathbf{N}^{-1} \mathbf{V} \boldsymbol{\Lambda} \mathbf{V}^* = (\mathbf{V}^*)^{-1} \boldsymbol{\Lambda} \mathbf{V}^*$ that $\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}_1^2$, where $\boldsymbol{\Sigma}_1^2$ is diagonal with eigenvalues of $\mathbf{A}^\sharp \mathbf{A}$ on the principal diagonal, $\sigma_{ii}^2 = \sigma_i^2$ for all $i = 1, \dots, n$. Since $\text{rank } \mathbf{L} = \text{rank } \mathbf{A} = r$, then the number of nonzero diagonal elements of $\boldsymbol{\Sigma}_1^2$ is equal r . Also, we note that

$$\begin{aligned} \mathbf{V}^* \mathbf{L} \mathbf{N}^{-1} \mathbf{V} &= \mathbf{V}^* \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-1} \mathbf{V} = \\ &= \mathbf{V}^{-1} \mathbf{A}^* (\mathbf{U}^*)^{-1} \mathbf{U}^{-1} \mathbf{A} (\mathbf{V}^*)^{-1} = \boldsymbol{\Sigma}_1^2. \end{aligned} \quad (12)$$

Consider the following matrix,

$$\mathbf{P} = \mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{V} \in \mathbb{H}^{m \times n}. \quad (13)$$

By virtue of (11),

$$\mathbf{P}^* \mathbf{P} = \left(\mathbf{V}^* \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M}^{\frac{1}{2}} \right) \mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{V} = \boldsymbol{\Sigma}_1^2. \quad (14)$$

Let us introduce the following $m \times n$ matrix $\mathbf{D} \in \mathbb{H}^{m \times n}$,

$$\mathbf{D} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (15)$$

where $\boldsymbol{\Sigma} \in \mathbb{H}^{r \times r}$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ on the principal diagonal. Then,

$$\mathbf{P} = \mathbf{M}^{\frac{1}{2}} \mathbf{U} \mathbf{D}. \quad (16)$$

By (13) and (16), we have $\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{V} = \mathbf{M}^{\frac{1}{2}} \mathbf{U} \mathbf{D}$. Due to the equality $(\mathbf{N}^{-1} \mathbf{V})^{-1} = \mathbf{V}^*$, it follows (10).

Now we shall prove (10), where σ_i^2 is the nonzero eigenvalues of $\mathbf{A} \mathbf{A}^\sharp = \mathbf{A} \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M}$. Since $\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*$ and \mathbf{M} are respectively Hermitian positive semidefinite and definite, then by Lemma 2.4 all eigenvalues of $\mathbf{A} \mathbf{A}^\sharp$ are nonnegative. Primarily, denote them by τ_i^2 , where $\tau_1 \geq \dots \geq \tau_m \geq 0$, and denote $\mathbf{Q} = \mathbf{A} \mathbf{A}^\sharp$. Since $\mathbf{M} \mathbf{Q} = \mathbf{M} \mathbf{A} \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M}$ is Hermitian and there exists a nonsingular $\mathbf{U} \in \mathbb{H}^{m \times m}$ such that $\mathbf{U}^* \mathbf{M} \mathbf{U} = \mathbf{I}_m$, then by Lemma 2.5,

$$\mathbf{U}^* \mathbf{M} \mathbf{Q} \mathbf{U} = \boldsymbol{\Omega}, \quad (17)$$

where \mathbf{U} is unitary with weight \mathbf{M} , and $\mathbf{\Omega}$ is a diagonal matrix.

It follows from $\mathbf{Q} = \mathbf{U}\mathbf{\Omega}\mathbf{U}^*\mathbf{M} = \mathbf{U}\mathbf{\Omega}\mathbf{U}^{-1}$ that $\mathbf{\Omega} \equiv \mathbf{\Sigma}_2^2$, where $\mathbf{\Sigma}_2^2$ is diagonal with eigenvalues of $\mathbf{A}\mathbf{A}^\sharp$ on the principal diagonal, $\tau_{ii}^2 = \tau_i^2$ for all $i = 1, \dots, m$. Since $\text{rank } \mathbf{Q} = \text{rank } \mathbf{A} = r$, then the number of nonzero diagonal elements of $\mathbf{\Sigma}_2^2$ is equal r . Also, we have

$$\begin{aligned} \mathbf{U}^*\mathbf{M}\mathbf{Q}\mathbf{U} &= \mathbf{U}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}\mathbf{U} = \\ &\mathbf{U}^{-1}\mathbf{A}(\mathbf{V}^*)^{-1}\mathbf{V}^{-1}\mathbf{A}^*(\mathbf{U}^*)^{-1} = \mathbf{\Sigma}_2^2. \end{aligned} \quad (18)$$

Comparing (12) and (18), and due to Lemma 2.2, we have that r nonzero eigenvalues of $\mathbf{A}\mathbf{A}^\sharp$ coincide with r nonzero eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$, i.e. $\sigma_i^2 = \tau_i^2$ for all $i = 1, \dots, r$.

Consider the following matrix,

$$\mathbf{S} = \mathbf{U}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \in \mathbb{H}^{m \times n}. \quad (19)$$

By virtue of (17),

$$\mathbf{S}\mathbf{S}^* = \mathbf{U}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{U} \right) = \mathbf{\Sigma}_2^2. \quad (20)$$

Consider again the matrix $\mathbf{D} \in \mathbb{H}^{m \times n}$ from (15). Then,

$$\mathbf{S} = \mathbf{D}\mathbf{V}^*\mathbf{N}^{-\frac{1}{2}}. \quad (21)$$

By (19) and (21), we have $\mathbf{U}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} = \mathbf{D}\mathbf{V}^*\mathbf{N}^{-\frac{1}{2}}$. From this, due to $(\mathbf{U}^*\mathbf{M})^{-1} = \mathbf{U}$, we again obtain (10). ■

Now, we prove the following theorem about a representation of $\mathbf{A}_{M,N}^\dagger$ by WSVD of $\mathbf{A} \in \mathbb{H}_r^{m \times n}$ with weights \mathbf{M} and \mathbf{N} .

Theorem 3.2 *Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, \mathbf{M} and \mathbf{N} be positive definite matrices of order m and n , respectively. There exist $\mathbf{U} \in \mathbb{H}^{m \times m}$, $\mathbf{V} \in \mathbb{H}^{n \times n}$ satisfying $\mathbf{U}^*\mathbf{M}\mathbf{U} = \mathbf{I}_m$ and $\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} = \mathbf{I}_n$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$, where $\mathbf{D} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Then the weighted Moore-Penrose inverse $\mathbf{A}_{M,N}^\dagger$ can be represented*

$$\mathbf{A}_{M,N}^\dagger = \mathbf{N}^{-1}\mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*\mathbf{M}, \quad (22)$$

where $\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and σ_i^2 is the nonzero eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$ or $\mathbf{A}\mathbf{A}^\sharp$, which coincide.

Proof. To prove the theorem we shall show that $\mathbf{X} = \mathbf{A}_{M,N}^\dagger$ expressed by (22) satisfies the equations (1), (2), (3N), and (4M).

$$1) \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^*\mathbf{N}^{-1}\mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^*\mathbf{M}\mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* = \mathbf{A},$$

$$2) \mathbf{XAX} = \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* \times \\ \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} = \mathbf{X},$$

$$(3M) (\mathbf{MAX})^* = \left(\mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} \right)^* = \\ \mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{m \times m} \mathbf{U}^* \mathbf{M} = \mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} = \\ \mathbf{MAX},$$

$$(4N) (\mathbf{NXA})^* = \left(\mathbf{N} \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* \right)^* = \\ \mathbf{N} \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{n \times n} \mathbf{V}^* = \mathbf{N} \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M} \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^* = \\ \mathbf{NXA}.$$

■

3.2 Limit representations of the weighted Moore-Penrose inverse over the quaternion skew field

Due to [3] the following limit representation can be extended to \mathbb{H} . We give the proof of the following lemma that different from ([3], Corollary 3.4.) and based on WSVD.

Lemma 3.1 *Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, and \mathbf{M} and \mathbf{N} be positive definite matrices of order m and n , respectively. Then*

$$\mathbf{A}_{M,N}^\dagger = \lim_{\lambda \rightarrow 0} (\lambda \mathbf{I} + \mathbf{A}^\sharp \mathbf{A})^{-1} \mathbf{A}^\sharp. \quad (23)$$

where $\mathbf{A}^\sharp = \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M}$, $\lambda \in \mathbb{R}_+$ and \mathbb{R}_+ is the set of all positive real numbers.

Proof. By Theorems 3.1 and 3.2, respectively, we have

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^*, \quad \mathbf{A}_{M,N}^\dagger = \mathbf{N}^{-1} \mathbf{V} \begin{pmatrix} \mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{U}^* \mathbf{M},$$

where $\mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and $\sigma_i^2 \in \mathbb{R}$ is the nonzero eigenvalues of $\mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A}$. Consider the matrix

$$\mathbf{D} := \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{D} = (\sigma_{ij}) \in \mathbb{H}_r^{m \times n}$ is such that $\sigma_{11} \geq \sigma_{22} \geq \dots \geq \sigma_{rr} > \sigma_{r+1, r+1} = \dots = \sigma_{qq} = 0$, $q = \min\{n, m\}$. Then

$$\mathbf{D}^* = \begin{pmatrix} \Sigma^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{D}^+ = \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$, $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M}$, $\mathbf{A}_{M,N}^\dagger = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^\dagger\mathbf{U}^*\mathbf{M}$. Since $\mathbf{N}^{-1}\mathbf{V} = (\mathbf{V}^*)^{-1}$, then we have

$$\begin{aligned} \lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A} &= \lambda\mathbf{I} + \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M}\mathbf{U}\mathbf{D}\mathbf{V}^* = \lambda\mathbf{I} + (\mathbf{V}^*)^{-1}\mathbf{D}^*\mathbf{D}\mathbf{V}^* = \\ &= (\mathbf{V}^*)^{-1}(\lambda\mathbf{I} + \mathbf{D}^2)\mathbf{V}^*. \end{aligned}$$

Further,

$$\begin{aligned} (\lambda\mathbf{I} + \mathbf{A}^\sharp\mathbf{A})^{-1}\mathbf{A}^\sharp &= (\mathbf{V}^*)^{-1}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{V}^*\mathbf{N}^{-1}\mathbf{V}\mathbf{D}^*\mathbf{U}^*\mathbf{M} = \\ &= \mathbf{N}^{-1}\mathbf{V}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D}^*\mathbf{U}^*\mathbf{M}. \end{aligned}$$

Consider the matrix

$$(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D} = \begin{pmatrix} \frac{\sigma_1}{\sigma_1^2 + \lambda} & \dots & 0 & & \\ \dots & \dots & \dots & & \mathbf{0} \\ 0 & \dots & \frac{\sigma_r}{\sigma_r^2 + \lambda} & & \vdots \\ & \vdots & & \ddots & \\ \mathbf{0} & & & & \mathbf{0} \end{pmatrix}.$$

It is obviously that $\lim_{\lambda \rightarrow 0} (\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D} = \mathbf{D}^\dagger$. Then,

$$\lim_{\lambda \rightarrow 0} \mathbf{N}^{-1}\mathbf{V}(\lambda\mathbf{I} + \mathbf{D}^2)^{-1}\mathbf{D}^*\mathbf{U}^*\mathbf{M} = \mathbf{N}^{-1}\mathbf{V}\mathbf{D}^\dagger\mathbf{U}^*\mathbf{M} = \mathbf{A}_{M,N}^\dagger.$$

The lemma is proofed. ■

In the following lemma we give another limit representation of $\mathbf{A}_{M,N}^\dagger$.

Lemma 3.2 *Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, and \mathbf{M} and \mathbf{N} be positive definite matrices of order m and n , respectively. Then*

$$\mathbf{A}_{M,N}^\dagger = \lim_{\lambda \rightarrow 0} \mathbf{A}^\sharp(\lambda\mathbf{I} + \mathbf{A}\mathbf{A}^\sharp)^{-1}, \quad (24)$$

where $\mathbf{A}^\sharp = \mathbf{N}^{-1}\mathbf{A}^*\mathbf{M}$, $\lambda \in \mathbb{R}_+$.

Proof. The proof is similar to the proof of Lemma 3.1 by using the fact from Theorem 3.1 that the nonzero eigenvalues of $\mathbf{A}^\sharp\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\sharp$ coincide. ■

It is evidently the following corollary.

Corollary 3.1 *If $\mathbf{A} \in \mathbb{H}^{m \times n}$, then the following statements are true.*

- i) *If $\text{rank } \mathbf{A} = n$, then $\mathbf{A}_{M,N}^\dagger = (\mathbf{A}^\sharp\mathbf{A})^{-1}\mathbf{A}^\sharp$.*
- ii) *If $\text{rank } \mathbf{A} = m$, then $\mathbf{A}_{M,N}^\dagger = \mathbf{A}^\sharp(\mathbf{A}\mathbf{A}^\sharp)^{-1}$.*
- iii) *If $\text{rank } \mathbf{A} = n = m$, then $\mathbf{A}_{M,N}^\dagger = \mathbf{A}^{-1}$.*

4 Determinantal representations of the weighted Moore-Penrose inverse over the quaternion skew field

Even though the eigenvalues of $\mathbf{A}^\sharp \mathbf{A}$ and $\mathbf{A} \mathbf{A}^\sharp$ are real and nonnegative, they are not Hermitian in general. Therefore, we consider two cases, when $\mathbf{A}^\sharp \mathbf{A}$ and $\mathbf{A} \mathbf{A}^\sharp$ both or one of them are Hermitian, and when they are non-Hermitian.

4.1 The case of Hermitian $\mathbf{A}^\sharp \mathbf{A}$ and $\mathbf{A} \mathbf{A}^\sharp$.

Let $(\mathbf{A}^\sharp \mathbf{A}) \in \mathbb{H}^{n \times n}$ be Hermitian. It means that $(\mathbf{A}^\sharp \mathbf{A})^* = (\mathbf{A}^\sharp \mathbf{A})$. Since \mathbf{N}^{-1} and \mathbf{M} are Hermitian, then

$$(\mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A})^* = \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-1} = \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A}.$$

So, to the matrix $(\mathbf{A}^\sharp \mathbf{A})$ be Hermitian the matrices \mathbf{N}^{-1} and $(\mathbf{A}^* \mathbf{M} \mathbf{A})$ should be commutative. Similarly, to $(\mathbf{A} \mathbf{A}^\sharp)$ be Hermitian the matrices \mathbf{M} and $(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)$ should be commutative.

Denote by $\mathbf{a}_{\cdot j}^\sharp$ and \mathbf{a}_i^\sharp the j -th column and the i -th row of \mathbf{A}^\sharp respectively.

Lemma 4.1 *If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then $\text{rank} (\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^\sharp) \leq r$.*

Proof. Let's lead elementary transformations of the matrix $(\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^\sharp)$ right-multiplying it by elementary unimodular matrices $\mathbf{P}_{i k}(-a_{j k})$, $k \neq j$. The matrix $\mathbf{P}_{i k}(-a_{j k})$ has $-a_{j k}$ in the (i, k) entry, 1 in all diagonal entries, and 0 in others. It is the matrix of an elementary transformation. Right-multiplying a matrix by $\mathbf{P}_{i k}(-a_{j k})$, where $k \neq j$, means adding to k -th column its i -th column right-multiplying on $-a_{j k}$. Then we get

$$(\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot j}^\sharp) \cdot \prod_{k \neq i} \mathbf{P}_{i k}(-a_{j k}) = \begin{pmatrix} \sum_{k \neq j} a_{1k}^\sharp a_{k1} & \dots & a_{1j}^\sharp & \dots & \sum_{k \neq j} a_{1k}^\sharp a_{kn} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{k \neq j} a_{nk}^\sharp a_{k1} & \dots & a_{nj}^\sharp & \dots & \sum_{k \neq j} a_{nk}^\sharp a_{kn} \end{pmatrix}_{i-th}.$$

The obtained matrix has the following factorization.

$$\begin{pmatrix} \sum_{k \neq j} a_{1k}^\sharp a_{k1} & \dots & a_{1j}^\sharp & \dots & \sum_{k \neq j} a_{1k}^\sharp a_{kn} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{k \neq j} a_{nk}^\sharp a_{k1} & \dots & a_{nj}^\sharp & \dots & \sum_{k \neq j} a_{nk}^\sharp a_{kn} \end{pmatrix}_{i-th} =$$

$$= \begin{pmatrix} a_{11}^\# & a_{12}^\# & \dots & a_{1m}^\# \\ a_{21}^\# & a_{22}^\# & \dots & a_{2m}^\# \\ \dots & \dots & \dots & \dots \\ a_{n1}^\# & a_{n2}^\# & \dots & a_{nm}^\# \end{pmatrix} \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{n1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & 0 & \dots & a_{mn} \end{pmatrix} j - th.$$

Denote by $\tilde{\mathbf{A}} := \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{n1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & 0 & \dots & a_{mn} \end{pmatrix} j - th$. The matrix $\tilde{\mathbf{A}}$ is obtained from \mathbf{A} by replacing all entries of the j -th row and of the i -th column with zeroes except that the (j, i) entry equals 1. Elementary transformations of a matrix do not change its rank and the rank of a matrix product does not exceed a rank of each factors. It follows that $\text{rank}(\mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{a}_{.j}^\#) \leq \min \{ \text{rank} \mathbf{A}^\#, \text{rank} \tilde{\mathbf{A}} \}$. It is obviously that $\text{rank} \tilde{\mathbf{A}} \geq \text{rank} \mathbf{A} = \text{rank} \mathbf{A}^\#$. This completes the proof. ■

The following lemma has been proved in the same way.

Lemma 4.2 *If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then $\text{rank}(\mathbf{A} \mathbf{A}^\#)_{.i}(\mathbf{a}_{.j}^\#) \leq r$.*

Analogues of the characteristic polynomial are considered in the following lemmas.

Lemma 4.3 *If $\mathbf{A} \in \mathbb{H}^{m \times n}$, $t \in \mathbb{R}$, and $(\mathbf{A}^\# \mathbf{A})$ is Hermitian, then*

$$\text{cdet}_i(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{a}_{.j}^\#) = c_1^{(ij)} t^{n-1} + c_2^{(ij)} t^{n-2} + \dots + c_n^{(ij)}, \quad (25)$$

where $c_n^{(ij)} = \text{cdet}_i(\mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{a}_{.j}^\#)$ and $c_k^{(ij)} = \sum_{\beta \in J_{k, n} \{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{a}_{.j}^\#))_{\beta}^{\beta}$ for all $k = \overline{1, n-1}$, $i = \overline{1, n}$, and $j = \overline{1, m}$.

Proof. Denote by $\mathbf{b}_{.i}$ the i -th column of the Hermitian matrix $\mathbf{A}^\# \mathbf{A} =: (b_{ij})_{n \times n}$. Consider the Hermitian matrix $(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{b}_{.i}) \in \mathbb{H}^{n \times n}$. It differs from $(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})$ an entry b_{ii} . Taking into account Lemma 2.3 we obtain

$$\det(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{.i}(\mathbf{b}_{.i}) = d_1 t^{n-1} + d_2 t^{n-2} + \dots + d_n, \quad (26)$$

where $d_k = \sum_{\beta \in J_{k, n} \{i\}} \det(\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta}$ is the sum of all principal minors of order k that contain the i -th column for all $k = \overline{1, n-1}$ and $d_n = \det(\mathbf{A}^\# \mathbf{A})$. Therefore,

we have

$$\mathbf{b}_{\cdot i} = \begin{pmatrix} \sum_l a_{1l}^\# a_{li} \\ \sum_l a_{2l}^\# a_{li} \\ \vdots \\ \sum_l a_{nl}^\# a_{li} \end{pmatrix} = \sum_l \mathbf{a}_{\cdot l}^\# a_{li},$$

where $\mathbf{a}_{\cdot l}^\#$ is the l th column-vector of $\mathbf{A}^\#$ for all $l = \overline{1, m}$. Taking into account Theorem 2.1, Remark 2.1 and Proposition 2.1 we obtain on the one hand

$$\begin{aligned} \det(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{b}_{\cdot i}) &= \text{cdet}_i(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{b}_{\cdot i}) = \\ &= \sum_l \text{cdet}_i(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{\cdot l}(\mathbf{a}_{\cdot l}^\# a_{li}) = \sum_l \text{cdet}_i(t\mathbf{I} + \mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot l}^\#) \cdot a_{li} \end{aligned} \quad (27)$$

On the other hand having changed the order of summation, we get for all $k = \overline{1, n-1}$

$$\begin{aligned} d_k &= \sum_{\beta \in J_{k, n} \{i\}} \det(\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta} = \sum_{\beta \in J_{k, n} \{i\}} \text{cdet}_i(\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta} = \\ &= \sum_{\beta \in J_{k, n} \{i\}} \sum_l \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot l}^\# a_{li}))_{\beta}^{\beta} = \\ &= \sum_l \sum_{\beta \in J_{k, n} \{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot l}^\#))_{\beta}^{\beta} \cdot a_{li}. \end{aligned} \quad (28)$$

By substituting (27) and (28) in (26), and equating factors at a_{li} when $l = j$, we obtain the equality (25). ■

By analogy can be proved the following lemma.

Lemma 4.4 *If $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $t \in \mathbb{R}$, and $\mathbf{A}\mathbf{A}^\#$ is Hermitian, then*

$$\text{rdet}_j(t\mathbf{I} + \mathbf{A}\mathbf{A}^\#)_{j \cdot}(\mathbf{a}_{\cdot i}^\#) = r_1^{(ij)} t^{n-1} + r_2^{(ij)} t^{n-2} + \dots + r_n^{(ij)},$$

where $r_n^{(ij)} = \text{rdet}_j(\mathbf{A}\mathbf{A}^\#)_{j \cdot}(\mathbf{a}_{\cdot i}^\#)$ and $r_k^{(ij)} = \sum_{\alpha \in I_{r, m} \{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^\#)_{j \cdot}(\mathbf{a}_{\cdot i}^\#))_{\alpha}^{\alpha}$

for all $k = \overline{1, n-1}$, $i = \overline{1, n}$, and $j = \overline{1, m}$.

The following theorem introduce the determinantal representations of the weighted Moore-Penrose by analogs of the classical adjoint matrix.

Theorem 4.1 *Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$. If $\mathbf{A}^\# \mathbf{A}$ or $\mathbf{A}\mathbf{A}^\#$ are Hermitian, then the weighted Moore-Penrose inverse $\mathbf{A}_{M, N}^\dagger = (\tilde{a}_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ possess the following determinantal representations, respectively,*

$$\tilde{a}_{ij}^\dagger = \frac{\sum_{\beta \in J_{r, n} \{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_{\cdot j}^\#))_{\beta}^{\beta}}{\sum_{\beta \in J_{r, n}} |(\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta}|}, \quad (29)$$

or

$$\tilde{a}_{ij}^\dagger = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j \left((\mathbf{A}\mathbf{A}^\#)_{j \cdot} (\mathbf{a}_i^\#) \right) \alpha}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^\#)_{\alpha}|}. \quad (30)$$

Proof. At first we prove (29). By Lemma 3.1, $\mathbf{A}^\dagger = \lim_{\alpha \rightarrow 0} (\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})^{-1} \mathbf{A}^\#$. Let $\mathbf{A}^\# \mathbf{A}$ is Hermitian. Then matrix $(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A}) \in \mathbb{H}^{n \times n}$ is a full-rank Hermitian matrix. Taking into account Theorem 2.6 it has an inverse, which we represent as a left inverse matrix

$$(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})^{-1} = \frac{1}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{n1} \\ L_{12} & L_{22} & \dots & L_{n2} \\ \dots & \dots & \dots & \dots \\ L_{1n} & L_{2n} & \dots & L_{nn} \end{pmatrix},$$

where L_{ij} is a left ij -th cofactor of $\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A}$. Then we have

$$\begin{aligned} & (\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})^{-1} \mathbf{A}^\# = \\ & = \frac{1}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} \begin{pmatrix} \sum_{k=1}^n L_{k1} a_{k1}^\# & \sum_{k=1}^n L_{k1} a_{k2}^\# & \dots & \sum_{k=1}^n L_{k1} a_{km}^\# \\ \sum_{k=1}^n L_{k2} a_{k1}^\# & \sum_{k=1}^n L_{k2} a_{k2}^\# & \dots & \sum_{k=1}^n L_{k2} a_{km}^\# \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^n L_{kn} a_{k1}^\# & \sum_{k=1}^n L_{kn} a_{k2}^\# & \dots & \sum_{k=1}^n L_{kn} a_{km}^\# \end{pmatrix}. \end{aligned}$$

Using the definition of a left cofactor, we obtain

$$\mathbf{A}_{M,N}^\dagger = \lim_{\alpha \rightarrow 0} \begin{pmatrix} \frac{\text{cdet}_1(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})_1(\mathbf{a}_1^\#)}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} & \dots & \frac{\text{cdet}_1(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})_1(\mathbf{a}_m^\#)}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} \\ \dots & \dots & \dots \\ \frac{\text{cdet}_n(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})_n(\mathbf{a}_1^\#)}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} & \dots & \frac{\text{cdet}_n(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})_n(\mathbf{a}_m^\#)}{\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})} \end{pmatrix}. \quad (31)$$

By Lemma 2.3, we have $\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A}) = \alpha^n + d_1 \alpha^{n-1} + d_2 \alpha^{n-2} + \dots + d_n$, where $d_k = \sum_{\beta \in J_{k,n}} |(\mathbf{A}^\# \mathbf{A})_{\beta}^\beta|$ is a sum of principal minors of $\mathbf{A}^\# \mathbf{A}$ of order k for all $k = \overline{1, n-1}$ and $d_n = \det \mathbf{A}^\# \mathbf{A}$. Since $\text{rank } \mathbf{A}^\# \mathbf{A} = \text{rank } \mathbf{A} = r$ and $d_n = d_{n-1} = \dots = d_{r+1} = 0$, it follows that

$$\det(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A}) = \alpha^n + d_1 \alpha^{n-1} + d_2 \alpha^{n-2} + \dots + d_r \alpha^{n-r}.$$

By using (25), we have

$$\text{cdet}_i(\alpha \mathbf{I} + \mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_j^\#) = c_1^{(ij)} \alpha^{n-1} + c_2^{(ij)} \alpha^{n-2} + \dots + c_n^{(ij)}$$

for all $i = \overline{1, n}$ and $j = \overline{1, m}$, where $c_k^{(ij)} = \sum_{\beta \in J_{k,n}\{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_j^\#))_{\beta}^\beta$ for all $k = \overline{1, n-1}$ and $c_n^{(ij)} = \text{cdet}_i(\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{a}_j^\#)$.

Now we prove that $c_k^{(ij)} = 0$, when $k \geq r + 1$ for all $i = \overline{1, n}$, and $j = \overline{1, m}$. By Lemma 4.1 rank $(\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp) \leq r$, then the matrix $(\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)$ has no more r right-linearly independent columns.

Consider $\left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta$, when $\beta \in J_{k,n}\{i\}$. It is a principal submatrix of $(\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)$ of order $k \geq r + 1$. Deleting both its i -th row and column, we obtain a principal submatrix of order $k - 1$ of $\mathbf{A}^\sharp \mathbf{A}$. We denote it by \mathbf{M} . The following cases are possible.

Let $k = r + 1$ and $\det \mathbf{M} \neq 0$. In this case all columns of \mathbf{M} are right-linearly independent. The addition of all of them on one coordinate to columns of $\left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta$ keeps their right-linear independence. Hence, they are basis in a matrix $\left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta$, and the i -th column is the right linear combination of its basic columns. From this by Theorem 2.4, we get $\text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta = 0$, when $\beta \in J_{k,n}\{i\}$ and $k \geq r + 1$.

If $k = r + 1$ and $\det \mathbf{M} = 0$, than p , ($p < k$), columns are basis in \mathbf{M} and in $\left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta$. Then due to Theorem 2.4, we obtain $\text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta = 0$ as well.

If $k > r + 1$, then by Theorem 2.5 it follows that $\det \mathbf{M} = 0$ and p , ($p < k - 1$), columns are basis in the both matrices \mathbf{M} and $\left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta$. Therefore, by Theorem 2.4, we obtain $\text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta = 0$.

Thus in all cases, we have $\text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta = 0$, when $\beta \in J_{k,n}\{i\}$ and $r + 1 \leq k < n$, and for all $i = \overline{1, n}$ and $j = \overline{1, m}$,

$$c_k^{(ij)} = \sum_{\beta \in J_{k,n}\{i\}} \text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta = 0,$$

$$c_n^{(ij)} = \text{cdet}_i (\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp) = 0.$$

Hence, $\text{cdet}_i (\alpha \mathbf{I} + \mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp) = c_1^{(ij)} \alpha^{n-1} + c_2^{(ij)} \alpha^{n-2} + \dots + c_r^{(ij)} \alpha^{n-r}$ for all $i = \overline{1, n}$ and $j = \overline{1, m}$. By substituting these values in the matrix from (31), we obtain

$$\mathbf{A}_{M,N}^+ = \lim_{\alpha \rightarrow 0} \begin{pmatrix} \frac{c_1^{(11)} \alpha^{n-1} + \dots + c_r^{(11)} \alpha^{n-r}}{\alpha^n + d_1 \alpha^{n-1} + \dots + d_r \alpha^{n-r}} & \dots & \frac{c_1^{(1m)} \alpha^{n-1} + \dots + c_r^{(1m)} \alpha^{n-r}}{\alpha^n + d_1 \alpha^{n-1} + \dots + d_r \alpha^{n-r}} \\ \dots & \dots & \dots \\ \frac{c_1^{(n1)} \alpha^{n-1} + \dots + c_r^{(n1)} \alpha^{n-r}}{\alpha^n + d_1 \alpha^{n-1} + \dots + d_r \alpha^{n-r}} & \dots & \frac{c_1^{(nm)} \alpha^{n-1} + \dots + c_r^{(nm)} \alpha^{n-r}}{\alpha^n + d_1 \alpha^{n-1} + \dots + d_r \alpha^{n-r}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{c_r^{(11)}}{d_r} & \dots & \frac{c_r^{(1m)}}{d_r} \\ \dots & \dots & \dots \\ \frac{c_r^{(n1)}}{d_r} & \dots & \frac{c_r^{(nm)}}{d_r} \end{pmatrix}.$$

$$\text{Here } c_r^{(ij)} = \sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp)\right)_\beta^\beta \text{ and } d_r = \sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp) \right|_\beta^\beta.$$

Thus, we have obtained the determinantal representation of $\mathbf{A}_{M,N}^+$ by (29). Similarly can be proved the determinantal representation of $\mathbf{A}_{M,N}^+$ by (30). ■

Remark 4.1 If $\text{rank } \mathbf{A} = n$, and $(\mathbf{A}^\# \mathbf{A})$ is Hermitian, then by Corollary 3.1, $\mathbf{A}_{M,N}^+ = (\mathbf{A}^\# \mathbf{A})^{-1} \mathbf{A}^\#$. Considering $(\mathbf{A}^\# \mathbf{A})^{-1}$ as a left inverse, we get the following representation of $\mathbf{A}_{M,N}^+$,

$$\mathbf{A}_{M,N}^+ = \frac{1}{\text{ddet} \mathbf{A}} \begin{pmatrix} \text{cdet}_1(\mathbf{A}^\# \mathbf{A})_{.1} (\mathbf{a}_{.1}^\#) & \dots & \text{cdet}_1(\mathbf{A}^\# \mathbf{A})_{.1} (\mathbf{a}_{.m}^\#) \\ \dots & \dots & \dots \\ \text{cdet}_n(\mathbf{A}^\# \mathbf{A})_{.n} (\mathbf{a}_{.1}^\#) & \dots & \text{cdet}_n(\mathbf{A}^\# \mathbf{A})_{.n} (\mathbf{a}_{.m}^\#) \end{pmatrix} \quad (32)$$

If $m > n$, then by Theorem 4.2 for $\mathbf{A}_{M,N}^+$ we have (29) as well.

Remark 4.2 If $\text{rank } \mathbf{A} = m$, and $(\mathbf{A} \mathbf{A}^\#)$ is Hermitian, then by Corollary 3.1, $\mathbf{A}_{M,N}^+ = \mathbf{A}^\# (\mathbf{A} \mathbf{A}^\#)^{-1}$. Considering $(\mathbf{A} \mathbf{A}^\#)^{-1}$ as a right inverse, we get the following representation of $\mathbf{A}_{M,N}^+$,

$$\mathbf{A}_{M,N}^+ = \frac{1}{\text{ddet} \mathbf{A}} \begin{pmatrix} \text{rdet}_1(\mathbf{A} \mathbf{A}^\#)_1 (\mathbf{a}_{1.}^\#) & \dots & \text{rdet}_m(\mathbf{A} \mathbf{A}^\#)_m (\mathbf{a}_{1.}^\#) \\ \dots & \dots & \dots \\ \text{rdet}_1(\mathbf{A} \mathbf{A}^\#)_1 (\mathbf{a}_{n.}^\#) & \dots & \text{rdet}_m(\mathbf{A} \mathbf{A}^\#)_m (\mathbf{a}_{n.}^\#) \end{pmatrix}. \quad (33)$$

If $m < n$, then by Theorem 4.2 for $\mathbf{A}_{M,N}^+$ we also have (30).

We obtain determinantal representations of the projection matrices $\mathbf{A}_{M,N}^+ \mathbf{A}$ and $\mathbf{A} \mathbf{A}_{M,N}^+$ in the following corollaries.

Corollary 4.1 If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, where $r < \min \{m, n\}$ or $r = m < n$, and $\mathbf{A}^\# \mathbf{A}$ is Hermitian, then the projection matrix $\mathbf{A}_{M,N}^+ \mathbf{A} =: \mathbf{P} = (p_{ij})_{n \times n}$ possess the following determinantal representation,

$$p_{ij} = \frac{\sum_{\beta \in J_{r, n} \setminus \{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{.i} (\mathbf{d}_{.j}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r, n}} |(\mathbf{A}^\# \mathbf{A})_{.i}^{\beta}|_{\beta}^{\beta}},$$

where $\mathbf{d}_{.j}$ is the j -th column of $\mathbf{A}^\# \mathbf{A} \in \mathbb{H}^{n \times n}$ and for all $i, j = \overline{1, n}$.

Proof. Representing \mathbf{A}^+ by (29) and right-multiplying it by \mathbf{A} , we obtain the following presentation of an entry p_{ij} of $\mathbf{A}_{M,N}^+ \mathbf{A} =: \mathbf{P} = (p_{ij})_{n \times n}$.

$$\begin{aligned}
p_{ij} &= \sum_k \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot k}^\sharp) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta} \right|} \cdot a_{kj} = \\
&= \frac{\sum_{\beta \in J_{r,n}\{i\}} \sum_k \text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{a}_{\cdot k}^\sharp) \right)_{\beta}^{\beta} \cdot a_{kj}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta} \right|} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{\cdot i} (\mathbf{d}_{\cdot j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta} \right|},
\end{aligned}$$

where $\mathbf{d}_{\cdot j}$ is the j -th column of $\mathbf{A}^\sharp \mathbf{A} \in \mathbb{H}^{n \times n}$ and for all $i, j = \overline{1, n}$. ■

By analogy can be proved the following corollary.

Corollary 4.2 *If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, where $r < \min\{m, n\}$ or $r = n < m$, and $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m \times m}$ is Hermitian, then the projection matrix $\mathbf{A}\mathbf{A}_{M,N}^\dagger =: \mathbf{Q} = (q_{ij})_{m \times m}$ possess the following determinantal representation,*

$$q_{ij} = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^\sharp)_{j \cdot} (\mathbf{g}_{i \cdot}))_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^\sharp)_{\alpha}^{\alpha}|},$$

where $\mathbf{g}_{i \cdot}$ is the i -th row of $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m \times m}$ and for all $i, j = \overline{1, m}$.

4.2 The case of non-Hermitian $\mathbf{A}^\sharp \mathbf{A}$ and $\mathbf{A}\mathbf{A}^\sharp$.

In this subsection we derive determinantal representations of the weighted Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m \times n}$, when $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m \times m}$ and $(\mathbf{A}^\sharp \mathbf{A}) \in \mathbb{H}^{n \times n}$ are non-Hermitian.

First, let $(\mathbf{A}^\sharp \mathbf{A}) \in \mathbb{H}^{n \times n}$ be non-Hermitian and $\text{rank}(\mathbf{A}^\sharp \mathbf{A}) < n$. Consider $(\lambda \mathbf{I} + \mathbf{A}^\sharp \mathbf{A})^{-1} \mathbf{A}^\sharp$. We have,

$$\begin{aligned}
(\lambda \mathbf{I} + \mathbf{A}^\sharp \mathbf{A})^{-1} \mathbf{A}^\sharp &= (\lambda \mathbf{I} + \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A})^{-1} \mathbf{A}^\sharp = (\mathbf{N}^{-1} (\lambda \mathbf{N} + \mathbf{A}^* \mathbf{M} \mathbf{A}))^{-1} \mathbf{A}^\sharp = \\
&= (\lambda \mathbf{N} + \mathbf{A}^* \mathbf{M} \mathbf{A})^{-1} \mathbf{A}^* \mathbf{M} = \mathbf{N}^{-\frac{1}{2}} (\lambda + \mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-\frac{1}{2}})^{-1} \mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} = \\
&= \mathbf{N}^{-\frac{1}{2}} \left(\lambda + \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^{-1} \left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M}^{\frac{1}{2}} \right) \mathbf{M}^{\frac{1}{2}} \quad (34)
\end{aligned}$$

Since by Lemma 2.1

$$\lim_{\lambda \rightarrow 0} \left(\lambda + \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^{-1} \left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M}^{\frac{1}{2}} \right) = \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^\dagger,$$

then combining (23) and (34), we obtain

$$\mathbf{A}_{M,N}^\dagger = \mathbf{N}^{-\frac{1}{2}} \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^\dagger \mathbf{M}^{\frac{1}{2}}. \quad (35)$$

Denote by \hat{a}_{ij} an ij -th entry of $\hat{\mathbf{A}}^\dagger := \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^\dagger$. By determinantal representing (8) for $\hat{\mathbf{A}}^\dagger$, we obtain

$$\hat{a}_{ij}^\dagger = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{.i} \left(\mathbf{m}^{\frac{1}{2}}\mathbf{a}\mathbf{n}^{-\frac{1}{2}} \right)^*_{.j} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{\beta}^{\beta} \right|} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{.i} \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^*\mathbf{m}^{\frac{1}{2}} \right)_{.j} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{\beta}^{\beta} \right|}$$

where $\left(\mathbf{m}^{\frac{1}{2}}\mathbf{a}\mathbf{n}^{-\frac{1}{2}}\right)^*_{.j}$ denote the j -th column of $\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}\right)^*$ for all $j = \overline{1, m}$. By $n_{ik}^{-\frac{1}{2}}$ denote an ik -th entry of $\mathbf{N}^{-\frac{1}{2}}$ for all $i, k = \overline{1, n}$, and by $m_{lj}^{\frac{1}{2}}$ denote an lj -th entry of $\mathbf{M}^{\frac{1}{2}}$ for all $l, j = \overline{1, m}$, respectively. Then for the weighted Moore-Penrose inverse $\mathbf{A}_{M,N}^+ = (\hat{a}_{ij}^\dagger) \in \mathbb{H}^{n \times m}$, we have

$$\begin{aligned} \tilde{a}_{ij}^\dagger &= \sum_k^n \sum_l^m n_{ik}^{-\frac{1}{2}} \hat{a}_{kl}^\dagger m_{lj}^{\frac{1}{2}} = \\ &= \frac{\sum_k n_{ik}^{-\frac{1}{2}} \cdot \sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_k \left(\left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{.k} \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^*\mathbf{m} \right)_{.j} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)^* \left(\mathbf{M}^{\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right) \right)_{\beta}^{\beta} \right|} = \\ &= \frac{\sum_k n_{ik}^{-\frac{1}{2}} \sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_k \left(\left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)_{.k} \left(\mathbf{n}^{-\frac{1}{2}}\mathbf{a}^*\mathbf{m} \right)_{.j} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left(\mathbf{N}^{-\frac{1}{2}}\mathbf{A}^*\mathbf{M}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \right)_{\beta}^{\beta} \right|}, \quad (36) \end{aligned}$$

for all $i = \overline{1, n}$, $j = \overline{1, m}$.

If $\text{rank}(\mathbf{A}^\sharp \mathbf{A}) = n$, then by Corollary 3.1, $\mathbf{A}_{M,N}^\dagger = (\mathbf{A}^\sharp \mathbf{A})^{-1} \mathbf{A}^\sharp$. So,

$$\mathbf{A}_{M,N}^\dagger = (\mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} \mathbf{A})^{-1} \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} = (\mathbf{A}^* \mathbf{M} \mathbf{A})^{-1} \mathbf{A}^* \mathbf{M}. \quad (37)$$

Since $\mathbf{A}^* \mathbf{M} \mathbf{A}$ is Hermitian, then we can use the determinantal representation of a Hermitian inverse matrix (7). Denote $\mathbf{A}^* \mathbf{M} =: (\hat{a})_{ij} \in \mathbb{H}^{n \times m}$. So, we have

$$\hat{a}_{ij}^\dagger = \frac{\sum_{k=1}^n L_{ki} \hat{a}_{kj}}{\det(\mathbf{A}^* \mathbf{M} \mathbf{A})} = \frac{\text{cdet}_i(\mathbf{A}^* \mathbf{M} \mathbf{A})_{.i} \hat{\mathbf{a}}_{.j}}{\det(\mathbf{A}^* \mathbf{M} \mathbf{A})}. \quad (38)$$

where $\hat{\mathbf{a}}_{.j}$ is the j -th column of $\mathbf{A}^* \mathbf{M}$ for all $j = \overline{1, m}$.

Now, let $(\mathbf{A}\mathbf{A}^\sharp) \in \mathbb{H}^{m \times m}$ be non-Hermitian and $\text{rank}(\mathbf{A}\mathbf{A}^\sharp) < m$. By determinantal representing (9) for $\hat{\mathbf{A}}^\dagger$, we similarly obtain

$$\hat{a}_{ij}^\dagger = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j \left(\left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right) \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \right)_j \left(\mathbf{n}^{-\frac{1}{2}} \mathbf{a}^* \mathbf{m}^{\frac{1}{2}} \right)_i \right)_\alpha}{\sum_{\alpha \in I_{r,m}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right) \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \right)_\alpha \right|},$$

where $\left(\mathbf{n}^{-\frac{1}{2}} \mathbf{a}^* \mathbf{m}^{\frac{1}{2}} \right)_i$ denote the i -th row of $\left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M}^{\frac{1}{2}} \right)$ for all $i = \overline{1, n}$. Finally, we get

$$\begin{aligned} \tilde{a}_{ij}^\dagger = & \sum_l \frac{\sum_{\alpha \in I_{r,m}\{l\}} \text{rdet}_l \left(\left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right) \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \right)_l \left(\mathbf{n}^{-1} \mathbf{a}^* \mathbf{m}^{\frac{1}{2}} \right)_i \right)_\alpha}{\sum_{\alpha \in I_{r,m}} \left| \left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right) \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)^* \right)_\alpha \right|} \cdot m_{lj}^{\frac{1}{2}} = \\ & \frac{\sum_l \sum_{\alpha \in I_{r,m}\{l\}} \text{rdet}_l \left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{A} \mathbf{M}^{\frac{1}{2}} \right)_l \left(\mathbf{n}^{-1} \mathbf{a}^* \mathbf{m}^{\frac{1}{2}} \right)_i \right)_\alpha m_{lj}^{\frac{1}{2}}}{\sum_{\alpha \in I_{r,m}} \left| \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{A} \mathbf{M}^{\frac{1}{2}} \right)_\alpha \right|}, \quad (39) \end{aligned}$$

for all $i = \overline{1, n}$, $j = \overline{1, m}$.

If $\text{rank}(\mathbf{A}\mathbf{A}^\sharp) = m$, then by Corollary 3.1, $\mathbf{A}_{M,N}^+ = \mathbf{A}^\sharp (\mathbf{A}\mathbf{A}^\sharp)^{-1}$. So,

$$\mathbf{A}_{M,N}^+ = \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M} (\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^* \mathbf{M})^{-1} = \mathbf{N}^{-1} \mathbf{A}^* (\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)^{-1}. \quad (40)$$

Since $\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*$ is Hermitian and full-rank, then we can use the determinantal representation of a Hermitian inverse matrix (6). Denote $\mathbf{N}^{-1} \mathbf{A}^* =: (\tilde{a})_{ij} \in \mathbb{H}^{n \times m}$. So, we have

$$\tilde{a}_{ij}^+ = \frac{\sum_{k=1}^n \hat{a}_{ik} R_{jk}}{\det(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)} = \frac{\text{rdet}_j(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)_j \cdot \hat{\mathbf{a}}_i}{\det(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)}, \quad (41)$$

where $\hat{\mathbf{a}}_i$ is the i -th row of $\mathbf{N}^{-1} \mathbf{A}^*$ for all $i = \overline{1, n}$.

Thus, we have proved the following theorem.

Theorem 4.2 *Let $\mathbf{A} \in \mathbb{H}_r^{m \times n}$. If $\mathbf{A}^\sharp \mathbf{A}$ is non-Hermitian, then the weighted Moore-Penrose inverse $\mathbf{A}_{M,N}^\dagger = (\tilde{a}_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ possess the determinantal representation (36) if $r < n$, and (38) if $r = n$. If $\mathbf{A}\mathbf{A}^\sharp$ is non-Hermitian, then $\mathbf{A}_{M,N}^\dagger = (\tilde{a}_{ij}^\dagger)$ possess the determinantal representation (39) if $r < m$, and (41) if $r = m$.*

Remark 4.3 *The equations (35), (37), and (40) expand the similarly well-known representations [23] of the weighted Moore-Penrose inverse to quaternion matrices.*

5 Determinantal representation of the weighted Moore-Penrose solution of system linear equations

Consider a right system linear equation over the quaternion skew field,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (42)$$

where $\mathbf{A} \in \mathbb{H}^{m \times n}$ is the coefficient matrix, $\mathbf{b} \in \mathbb{H}^{m \times 1}$ is a column of constants, and $\mathbf{x} \in \mathbb{H}^{n \times 1}$ is a unknown column. Due to [15] we have the following theorem that characterizes the weighted Moore-Penrose solution of (42).

Theorem 5.1 *The right system linear equation (42) with restriction $\mathbf{x} \in \mathcal{R}_r(\mathbf{A}^\sharp)$ has the unique solution $\tilde{\mathbf{x}} = \mathbf{A}_{M,N}^+ \mathbf{b}$.*

Theorem 5.2 *Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{A}^\sharp \mathbf{A} \in \mathbb{H}^{n \times n}$ be Hermitian.*

- (i) *If $\text{rank } \mathbf{A} = k \leq m < n$, then the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ of (42) possess the following determinantal representation*

$$\tilde{x}_i = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{f}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta}|}, \quad (43)$$

where $\mathbf{f} = \mathbf{A}^\sharp \mathbf{b}$, for all $i = \overline{1, n}$.

- (ii) *If $\text{rank } \mathbf{A} = n$, then for all $i = \overline{1, n}$ we have*

$$\tilde{x}_i = \frac{\text{cdet}_i(\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{f})}{\det \mathbf{A}^\sharp \mathbf{A}}. \quad (44)$$

Proof. i) If $\text{rank } \mathbf{A} = k \leq m < n$, then by Theorem 4.2 we can represent $\mathbf{A}_{M,N}^+$ by (29). By component-wise of $\tilde{\mathbf{x}} = \mathbf{A}_{M,N}^+ \mathbf{b}$, we have

$$\begin{aligned} \tilde{x}_i &= \sum_{j=1}^m \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta}|} \cdot b_j = \\ &= \frac{\sum_{\beta \in J_{r,n}\{i\}} \sum_j \text{cdet}_i((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{a}_{.j}^\sharp))_{\beta}^{\beta} \cdot b_j}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta}|} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^\sharp \mathbf{A})_{.i}(\mathbf{f}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^\sharp \mathbf{A})_{\beta}^{\beta}|}, \end{aligned}$$

where $\mathbf{f} = \mathbf{A}^\sharp \mathbf{b}$ and for all $i = \overline{1, n}$.

- ii) If $\text{rank } \mathbf{A} = n$, then $\mathbf{A}_{M,N}^+$ can be represented by (32). Representing $\mathbf{A}^+ \mathbf{b}$ by component-wise directly gives (44). ■

Theorem 5.3 Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{A}^\sharp \mathbf{A} \in \mathbb{H}^{n \times n}$ be non-Hermitian.

- (i) If $\text{rank } \mathbf{A} = k \leq m < n$, then the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ of (42) possess the following determinantal representation

$$\tilde{x}_i = \frac{\sum_k n_{ik}^{-\frac{1}{2}} \sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_k \left(\left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)_{\cdot k} \hat{\mathbf{f}} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} \mathbf{A} \mathbf{N}^{-\frac{1}{2}} \right)_{\beta}^{\beta} \right|},$$

where $\hat{\mathbf{f}} = \left(\mathbf{N}^{-\frac{1}{2}} \mathbf{A}^* \mathbf{M} \right) \mathbf{b}$ and $n_{ik}^{-\frac{1}{2}}$ is an ik -th entry of $\mathbf{N}^{-\frac{1}{2}}$ for all $i, k = \overline{1, n}$.

- (ii) If $\text{rank } \mathbf{A} = n$, then for all $i = \overline{1, n}$ we have

$$\tilde{x}_i = \frac{\text{cdet}_i(\mathbf{A}^* \mathbf{M} \mathbf{A})_{\cdot i} \check{\mathbf{f}}}{\det(\mathbf{A}^* \mathbf{M} \mathbf{A})}.$$

where $\check{\mathbf{f}} = \mathbf{A}^* \mathbf{M} \mathbf{b}$ for all $j = \overline{1, m}$.

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}_{M,N}^+$ by (36) in the (i) point, and (38) in the (ii) point, respectively. ■

Consider a left system linear equation over the quaternion skew field,

$$\mathbf{x} \mathbf{A} = \mathbf{b} \quad (45)$$

where $\mathbf{A} \in \mathbb{H}^{m \times n}$ is the coefficient matrix, $\mathbf{b} \in \mathbb{H}^{1 \times n}$ is a row of constants, and $\mathbf{x} \in \mathbb{H}^{1 \times m}$ is a unknown row. The following theorem characterizes the weighted Moore-Penrose solution of (45).

Theorem 5.4 The left system linear equation (45) with restriction $\mathbf{x} \in \mathcal{R}_l(\mathbf{A}^\sharp)$ has the unique solution $\tilde{\mathbf{x}} = \mathbf{b} \mathbf{A}_{M,N}^+$.

Theorem 5.5 Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{A} \mathbf{A}^\sharp \in \mathbb{H}^{m \times m}$ be Hermitian.

- (i) If $\text{rank } \mathbf{A} = k \leq n < m$, then the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_m)$ of (45) possess the following determinantal representation

$$\tilde{x}_j = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_i \left((\mathbf{A}^\sharp \mathbf{A})_{j \cdot} (\mathbf{g}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}^\sharp \mathbf{A})_{\alpha}^{\alpha}|},$$

where $\mathbf{g} = \mathbf{b} \mathbf{A}^\sharp$, for all $j = \overline{1, m}$.

- (ii) If $\text{rank } \mathbf{A} = m$, then for all $j = \overline{1, m}$ we have

$$\tilde{x}_j = \frac{\text{rdet}_j(\mathbf{A} \mathbf{A}^\sharp)_{j \cdot} (\mathbf{g})}{\det \mathbf{A} \mathbf{A}^\sharp}.$$

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}_{M,N}^+$ by (30) in the (i) point, and (33) in the (ii) point, respectively. ■

Theorem 5.6 *Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{A}\mathbf{A}^\# \in \mathbb{H}^{m \times m}$ be non-Hermitian.*

(i) *If $\text{rank } \mathbf{A} = k \leq n < m$, then the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_m)$ of (45) possess the following determinantal representation*

$$\tilde{x}_j = \frac{\sum_l \sum_{\alpha \in I_{r,m}\{l\}} \text{rdet}_l \left(\left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{A} \mathbf{M}^{\frac{1}{2}} \right)_l (\hat{\mathbf{g}})^\alpha \right) m_{lj}^{\frac{1}{2}}}{\sum_{\alpha \in I_{r,m}} \left| \left(\mathbf{M}^{\frac{1}{2}} \mathbf{A} \mathbf{N}^{-1} \mathbf{A} \mathbf{M}^{\frac{1}{2}} \right)_\alpha \right|},$$

where $\hat{\mathbf{g}} = \mathbf{b} \left(\mathbf{N}^{-1} \mathbf{A}^* \mathbf{M}^{\frac{1}{2}} \right)$, $m_{lj}^{\frac{1}{2}}$ is lj -th entry of $\mathbf{M}^{\frac{1}{2}}$ for all $l, j = \overline{1, m}$.

(ii) *If $\text{rank } \mathbf{A} = m$, then for all $j = \overline{1, m}$ we have*

$$\tilde{x}_j = \frac{\text{rdet}_j(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)_j \cdot \check{\mathbf{g}}}{\det(\mathbf{A} \mathbf{N}^{-1} \mathbf{A}^*)}.$$

where $\check{\mathbf{g}} = \mathbf{b} \mathbf{N}^{-1} \mathbf{A}^*$.

Proof. The proof is similar to the proof of Theorem 5.2 using component-wise representations of $\mathbf{A}_{M,N}^+$ by (39) in the (i) point, and (41) in the (ii) point, respectively. ■

6 Examples

In this section, we give examples to illustrate our results.

1. Let us consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & i & j \\ -k & i & 1 \\ k & j & -i \\ j & -1 & i \end{pmatrix}, \quad (46)$$

$$\mathbf{N}^{-1} = \begin{pmatrix} 23 & 16 - 2i - 2j + 10k & -16 + 10i - 2j - 2k \\ 16 + 2i + 2j - 10k & 29 & -19 - i - 13j - k \\ -16 - 10i + 2j + 2k & -19 + i + 13j + k & 29 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} 2 & k & i & 0 \\ -k & 2 & 0 & j \\ -i & 0 & 2 & k \\ 0 & -j & -k & 2 \end{pmatrix}. \quad (47)$$

By direct calculation we get that leading principal minors of \mathbf{M} and \mathbf{N}^{-1} are all positive. Therefore, due to Proposition 2.7, \mathbf{M} and \mathbf{N}^{-1} are positive definite matrices. Similarly, by direct calculation of leading principal minors of $\mathbf{A}^* \mathbf{A}$, we obtain $\text{rank } \mathbf{A}^* \mathbf{A} = \text{rank } \mathbf{A} = 2$.

Further,

$$\mathbf{A}^\# = \mathbf{M} \mathbf{A}^* \mathbf{N}^{-1} = \begin{pmatrix} 51 - 12i + 25j - 24k & -43 - 18i + 39k & -18 + 26i - 30j - 38k & 19 - i - 50j - 42k \\ -32i + 17j - 37k & -24 - 50i + 26j + 24k & -5 - 24i - 56j + k & -38 - 25i - 18j - 67k \\ 5 - 6i - 50j + 11k & 44 + 23i - 12j + 7k & 30 + 38i + 5j + 37k & 18 - 44i + 6j + 54k \end{pmatrix}.$$

Since,

$$\mathbf{A}^\# \mathbf{A} = \begin{pmatrix} 178 & 41 + 47i + 47j + 43k & -41 + 43i + 47j + 47k \\ 41 - 47i - 47j - 43k & 176 & -40 - 46i - 42j - 46k \\ -41 - 43i - 47j - 47k & -40 + 46i + 42j + 46k & 176 \end{pmatrix}$$

are Hermitian, then we shall be obtain $\mathbf{A}_{M,N}^+ = (\tilde{a}_{ij}^+) \in \mathbb{H}^{3 \times 4}$ due to Theorem 4.2 by Eq. (29).

We have, $\sum_{\beta \in J_{2,3}} \left| (\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta} \right| = 23380 + 23380 + 23380 = 70140$, and

$$\sum_{\beta \in J_{2,3}\{1\}} \text{cdet}_1 \left((\mathbf{A}^\# \mathbf{A})_{\cdot 1} (\mathbf{a}_{\cdot 1}^\#) \right)_{\beta}^{\beta} = 6680 + 1670i + 3340j - 5010k + 6680 - 5010i + 3340j - 1640k = 13360 - 3340i + 6680j - 6680k.$$

Then,

$$\tilde{a}_{11}^+ = \frac{8 - 2i + 4j - 4k}{42}.$$

Similarly, we obtain

$$\tilde{a}_{12}^+ = \frac{-7-3i+6k}{42}, \quad \tilde{a}_{13}^+ = \frac{-3+4i-5j-6k}{42}, \quad \tilde{a}_{14}^+ = \frac{3-8j-7k}{42},$$

$$\tilde{a}_{21}^+ = \frac{-5i+3j-6k}{42}, \quad \tilde{a}_{22}^+ = \frac{-4-8i+2j+2k}{42}, \quad \tilde{a}_{23}^+ = \frac{-1-4i-9j}{42}, \quad \tilde{a}_{24}^+ = \frac{-6-4i-3j-11k}{42},$$

$$\tilde{a}_{31}^+ = \frac{-1-i-8j+2k}{42}, \quad \tilde{a}_{32}^+ = \frac{7+4i-2j+k}{42}, \quad \tilde{a}_{33}^+ = \frac{5+6i+j+6k}{42}, \quad \tilde{a}_{34}^+ = \frac{3-7i+j+9k}{42}.$$

Finally, we obtain

$$\mathbf{A}_{M,N}^+ = \frac{1}{42} \begin{pmatrix} 8 - 2i + 4j - 4k & -7 - 3i + 6k & -3 + 4i - 5j - 6k & 3 - 8j - 7k \\ -5i + 3j - 6k & -4 - 8i + 2j + 2k & -1 - 4i - 9j & -6 - 4i - 3j - 11k \\ -1 - i - 8j + 2k & 7 + 4i - 2j + k & 5 + 6i + j + 6k & 3 - 7i + j + 9k \end{pmatrix}. \quad (48)$$

2. Consider the right system of linear equations,

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad (49)$$

where the coefficient matrix \mathbf{A} is (46) and the column $\mathbf{b} = (1 \ 0 \ i \ k)^T$. Using (48), by the matrix method we have for the weighted Moore-Penrose solution $\tilde{\mathbf{x}} = \mathbf{A}_{M,N}^+ \mathbf{b}$ of (49) with weights \mathbf{M} and \mathbf{N} from (47),

$$\tilde{x}_1 = \frac{11 - 13i - 2j + 4k}{42}, \tilde{x}_2 = \frac{15 - 9i + 7j - 3k}{42}, \tilde{x}_3 = \frac{-16 + 5i + 5j + 4k}{42}. \quad (50)$$

Now, we shall find the weighted Moore-Penrose solution of (49) by Cramer's rule (43). Since

$$\mathbf{f} = \mathbf{A}^\# \mathbf{b} = \begin{pmatrix} 67 - 80i - 12j + 25k \\ 91 - 55i + 43j - 19k \\ -97 + 30i + 31j + 24k \end{pmatrix},$$

then we have

$$\begin{aligned} \tilde{x}_1 &= \frac{\sum_{\beta \in J_{2,3}\{i\}} \text{cdet}_i((\mathbf{A}^\# \mathbf{A})_{\cdot i}(\mathbf{f}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^\# \mathbf{A})_{\beta}^{\beta}|} = \frac{18370 - 21710i - 3340j + 6680k}{70140} = \\ &\quad \frac{11 - 13i - 2j + 4k}{42}, \\ \tilde{x}_2 &= \frac{25050 - 15030i + 11690j - 5010k}{70140} = \frac{15 - 9i + 7j - 3k}{42}, \\ \tilde{x}_3 &= \frac{-26720 + 8350i + 8350j + 6680k}{70140} = \frac{-16 + 5i + 5j + 4k}{42}. \end{aligned}$$

As we expected, the weighted Moore-Penrose solutions by Cramer's rule and by the matrix method coincide.

7 Conclusion

In this paper, we derive determinantal representations of the weighted Moore-Penrose by WSVD within the framework of the theory of the noncommutative column-row determinants. Recently, within the framework of the theory of the noncommutative column-row determinants we have been obtain the determinantal representations of the Drazin inverse [24] and the weighted Drazin inverse [25] and corresponding determinantal representations of generalized inverse solutions of some matrix equations in [26–30].

References

- [1] R. Penrose, A generalized inverse for matrices, *Proc. Camb. Philos. Soc.* 51 (1955) 406–413.
- [2] K.M. Prasad, R.B. Bapat, A note of the Khatri inverse, *Sankhya: Indian J. Stat.* 54 (1992) 291-295.
- [3] Y. Wei, H. Wu, The representation and approximation for the weighted MoorePenrose inverse, *Appl. Math. Comput.* 121 (2001) 17-28.
- [4] I. V. Sergienko, E. F. Galba, V. S. Deineka, Limiting representations of weighted pseudoinverse matrices with positive definite weights. *Problem regularization, Cybernetics and Systems Analysis* 39(6) (2003) 816-830.
- [5] P. S. Stanimirovic', M. Stankovic', Determinantal representation of weighted Moore-Penrose inverse, *Mat. Vesnik* 46 (1994) 41-50.
- [6] X. Liu, G. Zhu, G. Zhou, Y. Yu, An analog of the adjugate matrix for the outer inverse $\mathbf{A}_{T,S}^{(2)}$, *Mathematical Problems in Engineering*, Volume 2012, Article ID 591256, 14 pages.
- [7] I.I. Kyrchei, Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules, *Linear and Multilinear Algebra* 56(4) (2008) 453-469.
- [8] X. Liu, Y. Yu, H. Wang, Determinantal representation of weighted generalized inverses, *Appl. Math. Comput.* 218(7) (2011) 3110-3121.
- [9] F. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* 251 (1997) 21-57.
- [10] I.I. Kyrchei, Determinantal representation of the MoorePenrose inverse matrix over the quaternion skew field, *Journal of Mathematical Sciences* 180(1) (2012) 23-33.
- [11] I. I. Kyrchei, The theory of the column and row determinants in a quaternion linear algebra, *Advances in Mathematics Research* 15, pp. 301-359. Nova Sci. Publ., New York, (2012).
- [12] I.I. Kyrchei Determinantal representations of the Moore-Penrose inverse over the quaternion skew field and corresponding Cramer's rules, *Linear and Multilinear Algebra* 59(4) (2011) 413-431.
- [13] C.F. Van Loan, Generalizing the singular value decomposition, *SIAM J. Numer. Anal.* 13 (1976) 76–83.
- [14] E. F. Galba, Weighted singular decomposition and weighted pseudoinversion of matrices, *Ukr. Math. J.* 48(10)(1996) 1618-1622.

- [15] G. Song, Q. Wang, H. Chang, Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field, *Comput Math. Appl.* 61 (2011) 1576-1589.
- [16] G.J. Song, Q.W. Wang, Condensed Cramer rule for some restricted quaternion linear equations, *Appl. Math. Comput.* 208(2) (2009) 556-563.
- [17] L. Huang, Wasin So, On left eigenvalues of a quaternionic matrix, *Linear Algebra Appl.* 323 (2001) 105-116.
- [18] W. So, Quaternionic left eigenvalue problem, *Southeast Asian Bulletin of Mathematics* 29 (2005) 555-565.
- [19] R. M. W. Wood, Quaternionic eigenvalues, *Bull. Lond. Math. Soc.* 17 (1985) 137-138.
- [20] A. Baker, Right eigenvalues for quaternionic matrices: a topological approach, *Linear Algebra and its Applications* 286 (1999) 303-309.
- [21] T. Dray, C. A. Manogue, The octonionic eigenvalue problem, *Advances in Applied Clifford Algebras* 8(2) (1998) 341-364.
- [22] R. A. Horn, C. R. Johnson, *Matrix Analysis*. Cambridge etc., Cambridge University Press, (1985).
- [23] A. Ben-Israel and T.N.E. Grenville. *Generalized Inverses: Theory and Applications*. Springer- Verlag, Berlin, (2002).
- [24] I. Kyrchei, Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations, *Appl. Math. Comput.* 238 (2014), pp. 193-207.
- [25] I. Kyrchei, Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. *Appl. Math. Comput.* 264 (2015) 453-465.
- [26] I. Kyrchei, Cramer's rule for quaternion systems of linear equations, *J. Math. Sci.* 155 (6) (2008) 839-858.
- [27] I. Kyrchei, Cramer's rule for some quaternion matrix equations, *Appl. Math. Comput.* 217(5) (2010) 2024-2030.
- [28] I. Kyrchei, Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations, *Appl. Math. Comput.* 219 (2013) 1576-1589.
- [29] I. Kyrchei, Analogs of Cramer's rule for the minimum norm least squares solutions of some matrix equations, *Appl. Math. Comput.* 218 (2012) 6375-6384.
- [30] I. Kyrchei, Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations, *Linear Algebra and Its Applications*, 438 no. 1, (2013), pp. 136152.